Oscillation Criteria of Impulsive Partial Difference Equations

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Abstract

In this paper, some oscillation criteria of certain impulsive partial difference equations with continuous variables are established.

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1 Introduction

The impulsive differential equations are form a mathematical apparatus for modelling of processes which at certain moments of their development undergo rapid changes. There are many good monographs on the impulsive differential equations[2, 3, 4, 6, 7]. Moreover partial difference equations arise from considerations of random walk problems, the study of molecular orbits, mathematical physics problems and finite-difference schemes. In the recent years, the investigation of the oscillation of partial difference equations with continuous variables has attracted more and more attantion in the literature, see e.g. [5, 8, 10, 11].
Let $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots$, be fixed points with $\lim_{n\to\infty} x_n = \infty$, and let for $n \in \mathbb{N}$ $x_{n+r} = x_n + \tau$, where r is a fixed natural number, and $\tau > 0$ is a constant. Define $J_{imp} = \{x_n\}_{n=1}^{\infty}$, $\mathbb{R}^+ = [0, \infty)$, $J = \{(x, y) : x \in J_{imp}, y \in \mathbb{R}^+\}$.
In 2009 Agarwal and Karakoc studied the oscillation of solutions of impulsive partial difference equations with continuous variables of the type

$$p_1 z(x+a, y+b)+p_2 z(x+a, y)+p_3 z(x, y+b)-p_4 z(x, y)+P(x, y)z(x-\tau, y-\sigma) = 0, \qquad (x, y) \in (\mathbb{R}^+\times\mathbb{R}^+)\setminus J,$$

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$$z(x_n^+, y) - z(x_n^-, y) = L_n z(x_n^-, y), \qquad (x_n, y) \in J,$$

where $z(x^+, y) = \lim_{\substack{(q,s)\to(x,y)\\ q>x}} z(q, s)$, $z(x^-, y) = \lim_{\substack{(q,s)\to(x,y)\\ q<x}} z(q, s)$[2].

This paper is concerned with the impulsive partial difference equations with continuous variables of the form

$$c_1u(x+a, y+b) + c_2u(x+a, y) + c_3u(x, y+b) - c_4u(x, y) + \sum_{i=1}^{\upsilon} P_i(x, y)u(x - \tau_i, y - \sigma_i) = 0, \quad (x, y) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus J, \tag{1}$$

$$u(x_n^+, y) - u(x_n^-, y) = L_n u(x_n^-, y), \qquad (x_n, y) \in J, \tag{2}$$

where $u(x^+, y) = \lim_{\substack{(q,s)\to(x,y)\\ q>x}} u(q, s)$, $u(x^-, y) = \lim_{\substack{(q,s)\to(x,y)\\ q<x}} u(q, s)$ and

$P_i \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+ - \{0\})$, a, b, τ_i, σ_i are positive constants, c_i, $i = 1, 2, 3, 4$, are nonnegative.
A function $u(x, y)$ in $[-\tau, \infty) \times [-\sigma, \infty)$ is said to be solution of (1)-(2) if

(i) for $(x, y) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus J$, u is continuous and satisfies (1),

(ii) for $(x, y) \in J$, $u(x^+, y)$ and $u(x^-, y)$ exist, $u(x^-, y) = u(x, y)$, and satisfy (2).

A solution $u(x, y)$ of (1)-(2) said to be eventually positive (or negative) if $u(x, y) > 0$ (or $u(x, y) < 0$) for all large x and y. It is said to be oscillatory if it is neither eventually positive nor eventually negative otherwise, it is called nonoscillatory.

2 Main Results

Throughout this paper we shall assume that

(A1) $\{L_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} L_n < \infty$,

(A2) $c_2, c_3 \geq c_4 > 0$,

(A3) $\tau_i = k_i a + \theta_i$, $\sigma_i = l_i b + \eta_i$, where k_i, l_i are nonnegative integers, $\theta_i \in [0, a)$ and $\eta_i \in [0, b)$.

(A4) $Q_i(x, y) = \min\{P_i(s, t) : x \leq s \leq x + a, y \leq t \leq y + b\}$
and

$$\inf Q_i(x, y) = q_i \geq 0, \quad i = 1, 2, \ldots, \upsilon$$

Lemma 1. *Assume that $u(x, y)$ be an eventually positive solution of* (1)-(2)*. Then for $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ the function*

$$\omega(x, y) = \int_x^{x+a} \int_y^{y+b} \Big(\prod_{x_0 < x_m < s} (1 + L_m)^{-1} \Big) u(s, t) \mathrm{d}t\mathrm{d}s \tag{3}$$

is an eventually positive solution of the partial difference inequality

$$c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y) \\ +\sum_{i=1}^{v}Q_i(x,y)\omega(x-k_ia,y-l_ib)\leq 0. \tag{4}$$

Here the symbol $\prod_{x_0<x_m<s}a_m$ *denotes the product of the members of the sequence* $\{a_m\}$ *over* m *such that* $x_m\in(x_0,s)\cap J_{imp}$*. If* $(x_0,s)\cap J_{imp}=\emptyset$*, or* $x_0\geq s$*, then we assume that* $\prod_{x_0<x_m<s}a_m=1$*.*

Proof. So that

$$\prod_{x_0<x_m<x}\big(1+L_m\big)^{-1}u(x,y)$$

is continuous for $(x,y)\in\mathbb{R}^+\times\mathbb{R}^+$,

$$\begin{aligned}\frac{\partial\omega}{\partial x}&=\int_y^{y+b}\Big[\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}u(x+a,t)-\prod_{x_0<x_m<x}\big(1+L_m\big)^{-1}u(x,t)\Big]\mathrm{d}t\\&=\int_y^{y+b}\prod_{x_0<x_m<x}\big(1+L_m\big)^{-1}\Big[\prod_{x\leq x_m<x+a}\big(1+L_m\big)^{-1}u(x+a,t)-u(x,t)\Big]\mathrm{d}t,\end{aligned}\tag{5}$$

$$\frac{\partial\omega}{\partial y}=\int_x^{x+a}\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}\Big[u(s,y+b)-u(s,y)\Big]\mathrm{d}s.\tag{6}$$

Since $u(x,y)$ is an eventually positive solution of (1)-(2),

$$c_1u(x+a,y+b)+c_2u(x+a,y)+c_3u(x,y+b)-c_4u(x,y)<0\tag{7}$$

for $(x,y)\in(\mathbb{R}^+\times\mathbb{R}^+)\setminus J$. From (A2) and (7)

$$u(x+a,y)-u(x,y)<0\qquad\text{and}\qquad u(x,y+b)-u(x,y)<0,$$

eventually. Moreover since $0<\prod_{x\leq x_m<x+a}\big(1+L_m\big)^{-1}\leq 1$ we obtain,

$$\prod_{x\leq x_m<x+a}\big(1+L_m\big)^{-1}u(x+a,y)-u(x,y)<0\tag{8}$$

Let $(x,y)\in J$ and say $x=x_n$. From (2) and (8), we have

$$\begin{aligned}u(x_n,y)&=\frac{1}{1+L_n}u(x_n^+,y)\\&\geq\frac{1}{1+L_n}\prod_{x_n^+\leq x_m<x_n^++a}\big(1+L_m\big)^{-1}u(x_n^++a,y)\\&=\prod_{x_n\leq x_m<x_n+a}\big(1+L_m\big)^{-1}u(x_n+a,y).\end{aligned}$$

Thus for all $(x,y)\in(\mathbb{R}^+\times\mathbb{R}^+)$ $\frac{\partial\omega}{\partial x}\le 0$ and $\frac{\partial\omega}{\partial y}\le 0$. Therefore

$$\begin{aligned}\omega(x-\tau,y-\sigma)=&\omega(x-(ka+\theta),y-(lb+\eta))\\ \ge&\omega(x-ka,y-lb).\end{aligned}\tag{9}$$

Since $0<\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\le 1$ from (1) we obtain

$$\begin{aligned}0=&c_1\int_x^{x+a}\int_y^{y+b}u(s+a,t+b)\mathrm{d}t\mathrm{d}s+c_2\int_x^{x+a}\int_y^{y+b}u(s+a,t)\mathrm{d}t\mathrm{d}s\\ &+c_3\int_x^{x+a}\int_y^{y+b}u(s,t+b)\mathrm{d}t\mathrm{d}s-c_4\int_x^{x+a}\int_y^{y+b}u(s,t)\mathrm{d}t\mathrm{d}s\\ &+\int_x^{x+a}\int_y^{y+b}\sum_{i=1}^{\upsilon}P_i(s,t)u(s-\tau_i,t-\sigma_i)\mathrm{d}t\mathrm{d}s\\ \ge&c_1\int_x^{x+a}\int_y^{y+b}\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s+a,t+b)\mathrm{d}t\mathrm{d}s\\ &+c_2\int_x^{x+a}\int_y^{y+b}\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s+a,t)\mathrm{d}t\mathrm{d}s\\ &+c_3\int_x^{x+a}\int_y^{y+b}\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s,t+b)\mathrm{d}t\mathrm{d}s-c_4\int_x^{x+a}\int_y^{y+b}u(s,t)\mathrm{d}t\mathrm{d}s\\ &+\int_x^{x+a}\int_y^{y+b}\sum_{i=1}^{\upsilon}P_i(s,t)\prod_{x_0<x_m<s}\big(1+L_m\big)^{-1}u(s-\tau_i,t-\sigma_i)\mathrm{d}t\mathrm{d}s.\end{aligned}\tag{10}$$

Using the definition of ω and Q_i from (9), (10) we have eventually

$$\begin{aligned}c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y)&\\ +\sum_{i=1}^{\upsilon}Q_i(x,y)\omega(x-k_ia,y-l_ib)\le 0.&\end{aligned}$$

Therefore $\omega(x,y)$ is an eventually positive solution of inequality (4). □

Since $\omega(x,y)>0$ from (4), for sufficiently large x and y, we have

$$c_2\omega(x+a,y)\le c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y)\quad,\quad c_3\omega(x,y+b)\le c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\omega(x,y).$$

Let $\lambda_1=0$. Then for sufficiently large x and y, we find

$$\omega(x-a,y)\ge e^{-\lambda_1}\frac{c_2}{c_4}\prod_{x_0<x_m<x}\big(1+L_m\big)^{-1}\omega(x,y)$$

$$\omega(x-k_ia,y)\ge e^{-k_i\lambda_1}\Big(\frac{c_2}{c_4}\Big)^{k_i}\prod_{j=1}^{k_i}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\omega(x,y)$$

and

$$\omega(x,y-b) \geq e^{-\lambda_1}\frac{c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\omega(x,y)$$

$$\omega(x,y-l_ib) \geq e^{-l_i\lambda_1}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y).$$

Therefore

$$\begin{aligned}\omega(x-k_ia,y-l_ib) &\geq e^{-k_i\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y-l_ib)\\ &\geq e^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y).\end{aligned}$$

From (4) and ((A4)) we obtain

$$\begin{aligned}c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b)-c_4\prod_{x_0<x_m<x+a}(1+L_m)\omega(x,y)\\ +\sum_{i=1}^{\upsilon}q_i\omega(x-k_ia,y-l_ib)\leq 0.\end{aligned} \tag{11}$$

Hence

$$\begin{aligned}c_2\omega(x+a,y) \leq c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\Big[1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\\ \times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\Big]\omega(x,y)\end{aligned}$$

and

$$\begin{aligned}c_3\omega(x,y+b) \leq c_4\prod_{x_0<x_m<x+a}\big(1+L_m\big)\Big[1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\\ \times\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\Big]\omega(x,y),\end{aligned}$$

eventually.
Let

$$e^{\lambda_2}=1-\frac{1}{c_4}\sum_{i=1}^{\upsilon}q_ie^{-(k_i+l_i)\lambda_1}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}.$$

It is clear that $\lambda_2 \le \lambda_1 = 0$.
Thus

$$\frac{c_2}{c_4} \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-1} \omega(x+a,y) \le e^{\lambda_2}\omega(x,y)$$

and

$$\frac{c_3}{c_4} \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-1} \omega(x,y+b) \le e^{\lambda_2}\omega(x,y),$$

eventually.
Hence

$$\omega(x-k_ia,y) \ge e^{-k_i\lambda_2}\big(\frac{c_2}{c_4}\big)^{k_i} \prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y)$$

and

$$\omega(x,y-l_ib) \ge e^{-l_i\lambda_2}\big(\frac{c_3}{c_4}\big)^{l_i} \prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y).$$

By induction, for $n \ge 1$,

$$\omega(x-a,y) \ge e^{-\lambda_n}\frac{c_2}{c_4}\prod_{x_0<x_m<x}\big(1+L_m\big)^{-1}\omega(x,y)$$

$$\omega(x-k_ia,y) \ge e^{-k_i\lambda_n}\big(\frac{c_2}{c_4}\big)^{k_i} \prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\omega(x,y)$$

and

$$\omega(x,y-b) \ge e^{-\lambda_n}\frac{c_3}{c_4}\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-1}\omega(x,y)$$

$$\omega(x,y-l_ib) \ge e^{-l_i\lambda_n}\big(\frac{c_3}{c_4}\big)^{l_i} \prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y),$$

where

$$e^{\lambda_n} = 1-\frac{1}{c_4}\sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i)\lambda_{n-1}}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}.$$

Remark that $-\infty < \lambda_n \le \lambda_{n-1} \le \ldots \le \lambda_1 = 0$. Hence $\lim_{n\to\infty}\lambda_n = \lambda^* < 0$ exists and

$$e^{\lambda^*} = 1-\frac{1}{c_4}\sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}. \tag{12}$$

Thus

$$\begin{aligned}\omega(x-k_ia,y-l_ib) \ge & e^{-(k_i+l_i)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\\ & \times \prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y)\end{aligned} \tag{13}$$

From (4)

$$c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big)\omega(x,y) \geq c_1\omega(x+a,y+b)+c_2\omega(x+a,y)+c_3\omega(x,y+b) + \sum_{i=1}^{\upsilon} Q_i(x,y)\omega(x-k_ia,y-l_ib). \tag{14}$$

From (13), we obtain

$$\omega(x-k_ia,y-l_ib) \geq e^{-(k_i+l_i-1)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i-1}\Big(\frac{c_3}{c_4}\Big)^{l_i} \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x-a,y)$$

and

$$\omega(x,y) \geq \frac{1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i-1}\Big(\frac{c_3}{c_4}\Big)^{l_i} \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\omega(x-a,y).$$

Therefore

$$\omega(x+a,y) \geq \frac{1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i-1}\Big(\frac{c_3}{c_4}\Big)^{l_i} \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-(l_i+1)}\omega(x,y).$$

By the same way we have

$$\omega(x,y+b) \geq \frac{1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i}\Big(\frac{c_3}{c_4}\Big)^{l_i-1} \times \prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y)$$

and

$$\omega(x+a,y+b) \geq \frac{1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i-1}\Big(\frac{c_3}{c_4}\Big)^{l_i-1} \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-l_i}\omega(x,y).$$

Substituting the above inequalities into (14), we obtain

$$
\begin{aligned}
c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big)\omega(x,y) \geq & \frac{c_1}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y+b)e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\
& \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-l_i}\omega(x,y) \\
& + \frac{c_2}{c_4}\sum_{i=1}^{\upsilon} Q_i(x+a,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \\
& \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-(l_i+1)}\omega(x,y) \\
& + \frac{c_3}{c_4}\sum_{i=1}^{\upsilon} Q_i(x,y+b)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\
& \times \prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x,y) \\
& + \sum_{i=1}^{\upsilon} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \\
& \times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x-a,y).
\end{aligned}
\tag{15}
$$

Thus

$$
\begin{aligned}
\Big(c_4 \prod_{x_0<x_m<x+a} (1+L_m) - &\frac{c_1}{c_4} \sum_{i=1}^{\upsilon} Q_i(x+a,y+b) e^{-(k_i+l_i-2)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i-1} \\
&\times \prod_{j=1}^{k_i-1} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+2a} \big(1+L_m\big)^{-l_i} \\
&- \frac{c_2}{c_4} \sum_{i=1}^{\upsilon} Q_i(x+a,y) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i} \\
&\times \prod_{j=1}^{k_i-1} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+2a} \big(1+L_m\big)^{-(l_i+1)} \\
&- \frac{c_3}{c_4} \sum_{i=1}^{\upsilon} Q_i(x,y+b) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i} \Big(\frac{c_3}{c_4}\Big)^{l_i-1} \\
&\times \prod_{j=1}^{k_i} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-l_i} \Big) \omega(x,y) \\
&\geq \sum_{i=1}^{\upsilon} Q_i(x,y) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i} \\
&\times \prod_{j=1}^{k_i-1} \Big(\prod_{x_0<x_m<x-ja} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-l_i} \omega(x-a,y).
\end{aligned}
$$

Set

$$
\begin{aligned}
R(x,y) = &c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big) - \frac{c_1}{c_4} \sum_{i=1}^{\upsilon} Q_i(x+a,y+b) e^{-(k_i+l_i-2)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i-1} \\
&\times \prod_{j=1}^{k_i-1} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+2a} \big(1+L_m\big)^{-l_i} \\
&- \frac{c_2}{c_4} \sum_{i=1}^{\upsilon} Q_i(x+a,y) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i-1} \Big(\frac{c_3}{c_4}\Big)^{l_i} \\
&\times \prod_{j=1}^{k_i-1} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+2a} \big(1+L_m\big)^{-(l_i+1)} \\
&- \frac{c_3}{c_4} \sum_{i=1}^{\upsilon} Q_i(x,y+b) e^{-(k_i+l_i-1)\lambda^*} \Big(\frac{c_2}{c_4}\Big)^{k_i} \Big(\frac{c_3}{c_4}\Big)^{l_i-1} \\
&\times \prod_{j=1}^{k_i} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-l_i}.
\end{aligned}
\tag{16}
$$

Then from (15), we obtain

$$\begin{aligned}\omega(x,y) \geq& \frac{1}{R(x,y)}\sum_{i=1}^{\upsilon} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \\ &\times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i}\omega(x-a,y).\end{aligned} \tag{17}$$

Similarly, we obtain

$$\begin{aligned}\omega(x,y) \geq& \frac{1}{R(x,y)}\sum_{i=1}^{\upsilon} Q_i(x,y)e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\ &\times \prod_{j=1}^{k_i}\Big(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i-1)}\omega(x,y-b)\end{aligned} \tag{18}$$

and

$$\begin{aligned}\omega(x,y) \geq& \frac{1}{R(x,y)}\sum_{i=1}^{\upsilon} Q_i(x,y)e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\ &\times \prod_{j=1}^{k_i-1}\Big(\prod_{x_0<x_m<x-ja}\big(1+L_m\big)^{-1}\Big)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i-1)}\omega(x-a,y-b).\end{aligned} \tag{19}$$

From (14), we have

$$1 \geq \frac{c_1}{c_4 \prod_{x_0<x_m<x+a}\big(1+L_m\big)} \frac{\omega(x+a,y+b)}{\omega(x,y)} + \frac{c_2}{c_4 \prod_{x_0<x_m<x+a}\big(1+L_m\big)} \frac{\omega(x+a,y)}{\omega(x,y)}$$
$$+ \frac{c_3}{c_4 \prod_{x_0<x_m<x+a}\big(1+L_m\big)} \frac{\omega(x,y+b)}{\omega(x,y)} + \frac{1}{c_4 \prod_{x_0<x_m<x+a}\big(1+L_m\big)} \sum_{i=1}^{\upsilon} Q_i(x,y) \frac{\omega(x-k_i a, y-l_i b)}{\omega(x,y)}$$
$$\geq \frac{c_1}{c_4} \frac{1}{R(x+a,y+b)} \sum_{i=1}^{\upsilon} Q_i(x+a,y+b) e^{-(k_i+l_i-2)\lambda^*} \big(\frac{c_2}{c_4}\big)^{k_i-1} \big(\frac{c_3}{c_4}\big)^{l_i-1}$$
$$\times \prod_{j=1}^{k_i} \Big(\prod_{x_0<x_m<x-(j-2)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+2a} \big(1+L_m\big)^{-(l_i-1)}$$
$$+ \frac{c_2}{c_4} \frac{1}{R(x+a,y)} \sum_{i=1}^{\upsilon} Q_i(x+a,y) e^{-(k_i+l_i-1)\lambda^*} \big(\frac{c_2}{c_4}\big)^{k_i-1} \big(\frac{c_3}{c_4}\big)^{l_i}$$
$$\times \prod_{j=1}^{k_i} \Big(\prod_{x_0<x_m<x-(j-2)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+2a} \big(1+L_m\big)^{-l_i}$$
$$+ \frac{c_3}{c_4} \frac{1}{R(x,y+b)} \sum_{i=1}^{\upsilon} Q_i(x,y+b) e^{-(k_i+l_i-1)\lambda^*} \big(\frac{c_2}{c_4}\big)^{k_i} \big(\frac{c_3}{c_4}\big)^{l_i-1}$$
$$\times \prod_{j=1}^{k_i} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-l_i}$$
$$+ \frac{1}{c_4} \sum_{i=1}^{\upsilon} Q_i(x,y) e^{-(k_i+l_i)\lambda^*} \big(\frac{c_2}{c_4}\big)^{k_i} \big(\frac{c_3}{c_4}\big)^{l_i}$$
$$\times \prod_{j=1}^{k_i} \Big(\prod_{x_0<x_m<x-(j-1)a} \big(1+L_m\big)^{-1} \Big) \prod_{x_0<x_m<x+a} \big(1+L_m\big)^{-(l_i+1)} = T(x,y), \tag{20}$$

for all large x and y.
From (20), we obtain the following main result.

Theorem 1. *Suppose that*

$$\limsup_{x,y\to\infty} T(x,y) > 1. \tag{21}$$

Then every solution of (1)-(2) *is oscillatory.*

From (12), we obtain for all large x and y,

$$e^{\lambda^*} = 1 - \frac{1}{c_4} \sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i)\lambda^*} \big(\frac{c_2}{c_4}\big)^{k_i} \big(\frac{c_3}{c_4}\big)^{l_i} L^{-(k_i+l_i+1)},$$

where $L = \prod_{m=1}^{\infty} \big(1+L_m\big)$.
Therefore

$$c_4 L\big(1-e^{\lambda^*}\big) = \sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i)\lambda^*} \big(\frac{c_2}{c_4}\big)^{k_i} \big(\frac{c_3}{c_4}\big)^{l_i} L^{-(k_i+l_i)}, \tag{22}$$

eventually.
By using (16) and (22), we have, for all large x and y,

$$\begin{aligned}
R(x,y) \leq & c_4 \prod_{x_0<x_m<x+a} \big(1+L_m\big) - \frac{c_1}{c_4}\sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i-2)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\
& \times \prod_{j=1}^{k_i-1}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-l_i} \\
& -\frac{c_2}{c_4}\sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i-1}\big(\frac{c_3}{c_4}\big)^{l_i} \\
& \times \prod_{j=1}^{k_i-1}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\prod_{x_0<x_m<x+2a}\big(1+L_m\big)^{-(l_i+1)} \\
& -\frac{c_3}{c_4}\sum_{i=1}^{\upsilon} q_i e^{-(k_i+l_i-1)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i-1} \\
& \times \prod_{j=1}^{k_i}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-l_i} \\
\leq & c_4 L - \frac{c_1}{c_4}c_4 L^2\big(1-e^{\lambda^*}\big)e^{2\lambda^*}\frac{c_4^2}{c_2c_3} \\
& -\frac{c_2}{c_4}c_4L\big(1-e^{\lambda^*}\big)e^{\lambda^*}\frac{c_4}{c_2} - \frac{c_3}{c_4}c_4L\big(1-e^{\lambda^*}\big)e^{\lambda^*}\frac{c_4}{c_3} \\
= & c_4 L\Bigg[1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\Bigg(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Bigg)\Bigg].
\end{aligned}$$

Hence

$$\begin{aligned}
T(x,y) \geq & \frac{1}{c_4 L\Bigg[1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\Bigg(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Bigg)\Bigg]}\Bigg\{\frac{c_1}{c_4}c_4L^2\big(1-e^{\lambda^*}\big)e^{2\lambda^*}\frac{c_4^2}{c_2c_3} \\
& +\frac{c_2}{c_4}c_4L\big(1-e^{\lambda^*}\big)e^{\lambda^*}\frac{c_4}{c_2}+\frac{c_3}{c_4}c_4L\big(1-e^{\lambda^*}\big)e^{\lambda^*}\frac{c_4}{c_3}\Bigg\} + \frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i)\lambda^*} \\
& \times\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i}\prod_{j=1}^{k_i}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)} \qquad (23) \\
= & \frac{\big(1-e^{\lambda^*}\big)e^{\lambda^*}\Bigg(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Bigg)}{1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\Bigg(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\Bigg)} + \frac{1}{c_4}\sum_{i=1}^{\upsilon}Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\big(\frac{c_2}{c_4}\big)^{k_i}\big(\frac{c_3}{c_4}\big)^{l_i} \\
& \times\prod_{j=1}^{k_i}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}
\end{aligned}$$

Corollary 1. If

$$\begin{aligned}\limsup_{x,y\to\infty}\frac{1}{c_4}\sum_{i=1}^{v}Q_i(x,y)e^{-(k_i+l_i)\lambda^*}\Big(\frac{c_2}{c_4}\Big)^{k_i}\Big(\frac{c_3}{c_4}\Big)^{l_i}\\ \times\prod_{j=1}^{k_i}\Bigg(\prod_{x_0<x_m<x-(j-1)a}\big(1+L_m\big)^{-1}\Bigg)\prod_{x_0<x_m<x+a}\big(1+L_m\big)^{-(l_i+1)}\\ >\frac{1-2\big(1-e^{\lambda^*}\big)e^{\lambda^*}\left(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\right)}{1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\left(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\right)},\end{aligned} \tag{24}$$

then every solution of (1)-(2) is oscillatory.

From (20),

$$T(x,y)\geq\frac{\big(1-e^{\lambda^*}\big)e^{\lambda^*}\left(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\right)}{1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\left(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\right)}+\big(1-e^{\lambda^*}\big), \tag{25}$$

for all large x and y.

Corollary 2. If

$$\frac{\big(1-e^{\lambda^*}\big)e^{\lambda^*}\left(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\right)}{1-\big(1-e^{\lambda^*}\big)e^{\lambda^*}\left(\frac{c_1c_4}{c_2c_3}Le^{\lambda^*}+2\right)}+\big(1-e^{\lambda^*}\big)>1, \tag{26}$$

then every solution of (1)-(2) is oscillatory.

Remark that the special case of (1): $v=1$, $k=l=0$, $c_1=c_2=c_3=c_4=0$. From Theorem 9 of [2], if

$$\limsup_{x,y\to\infty}Q(x,y)>L$$

then every solution of (1)-(2) is oscillatory.
In this case, $e^{\lambda^*}=1-\frac{q}{L}$. By Corollary 1, if

$$\limsup_{x,y\to\infty}Q(x,y)>L\frac{1-2\frac{q}{L}\big(1-\frac{q}{L}\big)\big(L-q+2\big)}{1-\frac{q}{L}\big(1-\frac{q}{L}\big)\big(L-q+2\big)}, \tag{27}$$

then every solution of (1)-(2) is oscillatory.
It is clear that the right-hand side of (27) is less than L. So our result is sharper than that in [2] in this case.

References

[1] R.P. Agarwal. *Difference Equations and Inequalities*. Marcel Dekker, New York, 1992.

[2] R.P. Agarwal, F. Karakoc. Oscillation of impulsive partial difference equations with continuous variables. *Math. Comput. Modelling*, vol. 50, pp. 1262–1278, (2009).

[3] D.D. Baĭnov, M.B. Dimitrova, A.B. Dishlies. Oscillation of bounded solutions of impulsive differential–difference of second order. *Appl. Math. Comput.*, vol. 114, pp. 61–68, (2000).

[4] D.D. Baĭnov, P.S. Simeonov. *Impulsive Differential Equations Asymtotic Properties of the Solutions*. World Scientific, Singapore, 1995.

[5] B.T. Cui, Y. Liu. Oscillatory for partial difference equations with continuous variables. *J. Comput. Appl. Math.*, vol. 154, pp. 373–391, (2003).

[6] V. Lakshmikantham, D.D. Baĭnov and P.S. Simeonov. *Theory of Impulsive Differential Equations*. World Scientific, Singapore, 1998.

[7] E. Minchev. Oscillation of solutions of impulsive nonlinear hyperbolic differential–difference equations. *Math. Balcanica*, vol. 12, no. 1–2, pp.215–224 , (1998).

[8] B.G. Zhang. Oscillation criteria of partial difference equations with continuous variables. *Acta Math. Sinica*, vol. 42, no. 3, pp. 487–494, (1999) (in Chinese).

[9] B.G. Zhang, B.M. Liu. Oscillation criteria of certain nonlinear partial difference equations. *Comput. Math. Appl.*, vol. 38, pp. 107–112, (1999).

[10] B.G. Zhang, B.M. Liu. Necessary and sufficient conditions for oscillation of partial difference equations with continuous variables. *Comput. Math. Appl.*, vol. 38, pp. 163–167, (1999).

[11] B.G. Zhang, Y.H. Wang. Oscillation theorems for certain delay partial difference equation. *Appl. Math. Letters*, vol. 19, pp. 639–646, (2006).

[12] B.G. Zhang, Y. Zhou. *Qualitative Analysis of Delay Partial Difference Equations*. Hindawi Publishing Corporation, New York, 2007.